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Abstract

We consider a school choice problem where schools' priorities depend on transferable students' characteristics. A school choice algorithm selects for each profile of students' preferences over schools an assignment of students to schools and a final allocation of characteristics (an extended matching). We define the Student Exchange with Transferable Characteristics (SETC) class of algorithms. Each SETC always selects a constrained efficient extended matching. That is an extended matching that i) is stable according to the priorities generated by the final allocation of characteristics and ii) is not Pareto dominated by another stable extended matching. Every constrained efficient extended matching that Pareto improves upon a stable extended matching can be obtained via an algorithm in the SETC class. When students' characteristics are fully transferable, a specific algorithm in the SETC family is equivalent to the application of the Top Trade Cycle Algorithm starting from the Student Optimal Stable Matching.

Keywords: School Choice, Transferable Characteristics,Priorities, Constrained Efficiency.

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School Choice with Transferable Students' Characteristics*

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We consider a school choice problem where schools' priorities depend on transferable students' characteristics. We define the Student Exchange with Transferable Characteristics (SETC) class of algorithms. Each SETC always selects a constrained efficient extended matching. That is an extended matching that i) is stable according to the priorities generated by the final allocation of characteristics and ii) is not Pareto dominated by another stable extended matching. Every constrained efficient extended matching that Pareto improves upon a stable extended matching can be obtained via an algorithm in the SETC class. When students' characteristics are fully transferable, a specific algorithm in the SETC family is equivalent to the application of the Top Trade Cycle Algorithm starting from the Student Optimal Stable Matching.

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1 Introduction

School choice programs consider systematic procedures for assigning students to schools. Each student submits a list of preferences of schools to a central placement authority, such as the school district. This central authority, then decides to which students will attend each school. An algorithm (or mechanism) selects a matching of students to schools considering students' reported preferences over schools and on schools' priority rankings that determine who will get a seat in case a school is over demanded. A major concern in the design of school choice programs has been to secure the selection of fair matchings of students to schools. That is, all the students that obtain a seat at a given school should have a higher priority at that school than the students who preferred that school rather than the one they are matched to. In the recent years, a vast majority of school districts have implemented school choice algorithms based on Gale and Shapley's Deferred Acceptance Algorithm (DA)(Gale and Shapley, 1962; Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005; Pathak, 2016). The application of DA with proposing students always selects a stable matching, that is a fair, individually rational, and non-wasteful matching.[1]

In the canonical school choice problem priorities are a primitive of the model (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). School districts use several criteria in determining a priority order for a school based on different characteristics of the potential students as walk-zone, number of siblings in the school, and other socio-economic characteristics. Arbitrary tie-breaking criteria are often necessary to solve ties when students share the same characteristics.[2]

We propose a new model that relies on students' characteristics as primitives instead of schools' priorities. Students are initially endowed with characteristics specific to each school. Each school priorities are defined over pairs of students and students' specific

[1]A matching is individually rational if no student is assigned to a school that she rather not to attend. A matching is non-wasteful if for each student every school that she prefers to the school she is assigned to has filled all her available seats.

[2]In Boston, neighborhood walk-zone and having siblings in the school are the characteristics that determine schools' priorities(Abdulkadiroğlu et al., 2005). In Spain, characteristics as family income, number of siblings, siblings attending the school, or "legacy" points if her parents or older siblings attended that school are also considered. See Górtazar et al. (2020); Casalmiglia et al. (2020) for detailed descriptions of Barcelona and Madrid priority systems.

characteristics for that school. Students can exchange the characteristics of different schools and affect to their position in the priority ranking of those schools. In this context, a matching that Pareto improves an initial matching but may not respect fairness under the priority rankings of students generated by the initial distribution of characteristics, but it may be admissible on fairness grounds after an exchange of the relevant characteristics among the improving students.[3]

We propose a class of school choice algorithms, Student Exchange with Transferable Characteristics (SETC). Each algorithm in this class selects a *constrained efficient extended matching*. That is, the outcome of a SETC algorithm is a matching of students to schools and a redistribution of specific transferable characteristics such that i) the matching is stable with respect to schools' priorities for the new distribution of characteristics, and ii) the matching is not Pareto dominated by another stable matching with respect to further admissible redistributions of characteristics. Every such constrained efficient extended matching can be obtained by an algorithm in the SETC class (Theorem 1). Moreover, a specific algorithm in the SETC family, the Top Trade SETC algorithm, that replicates the idea behind the application of Gale's Top Trade Cycle Mechanism (TTCM) *à la* Shapley and Scarf (1974) is well defined when school priorities are monotonous and fully transferable (Theorem 2).

Our approach can be useful to improve the efficiency in situations where in the formation of priorities we can distinguish between allocative criteria (as the tiebreaker) and fairness constraints (as the need for siblings to attend the same school). For example, we can think about the allocation of medical resources. Consider a situation where public medical services are regionally managed as in Spain. In this case a patient may be eligible for a different treatment or drug depending on her characteristics (like age, life expectancy

[3] An obvious candidate for such a transferable characteristic would be the exchange of the tie-breaking lottery draws at different schools when different schools use different tie-breaking criteria. Multiple tie-breaking criteria are justified since they reduce the chances that over-demanded schools systematically reject a student that has a bad lottery draw (Arnosti, 2016). The Amsterdam's school choice program reform in 2014 introduced a system based on DA with multiple tie-breaking criteria. In 2015 this decision was challenged in court by families who wished to switch school seats that could be justified with an exchange of the priorities obtained with multiple tie-breaking criteria. The issue is discussed in Ashlagi et al. (2019) and https://www.nemokennislink.nl/publicatiesschoolstrijd-in-amsterdam/ (Schoolstrijd in Amsterdam) (Arnout Jaspers, Kennislink, July 1, 2015, accessed May 21st 2020).

or health) and on the bundle of treatments acquired by her regional government. If we consider that patients' characteristics are determinant to access treatment and the patient residence only as an allocative criterion the algorithms in the SETC class allow us to implement welfare improving exchanges of treatments or drugs across regions while respecting fairness requirements.

1.1 Related Literature

The school choice problem was first presented by Balinski and Sönmez (1999) that introduce the idea of fairness to allocate seats to students. Abdulkadiroğlu and Sönmez (2003) analyze the problem from a mechanism design perspective. The authors show that the students proposing DA always selects stable matchings and it is strategy-proof.[4] They also study an adaptation of the Gale's TTCM (Shapley and Scarf, 1974) and show that it always selects Pareto efficient matchings and it is strategy-proof. Unfortunately, stable matchings are not efficient and the level of inefficiency of a stable matching can be severe (Dur and Morrill, 2017; Kesten, 2010; Abdulkadiroğlu et al., 2009)

There have been several attempts to alleviate the conflict between stability and efficiency by weakening the notion of fairness. Kesten (2010) proposes the Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) that finds a constrained efficient matching by analyzing the possibility that students consent to renounce to their priorities over schools where they cannot obtain a seat under the students proposing DA.[5] Alcalde and Romero-Medina (2017) propose an alternative weakening of fairness: α-equitability. Ehlers and Morrill (2019) relax the fairness constraint and propose a stable set of legal matchings that are not dominated in fairness terms by any other legal matching. Finally, Alva and Manjunath (2019) present the concept of stable domination.

The paper closer to ours is Dur et al. (2019) that proposes an alternative weakening of stability, partial stability. Under partial stability some priorities of some students at some schools can be ignored. Then it explores the welfare gains that can be captured by applying the improvement cycles proposed by Erdil and Ergin (2008). Our paper shares with Dur et al. (2019) the use of improvement cycles. Beyond this point both papers have considerable differences. First, the primitives in our model are not the schools' priorities

[4] A mechanism is strategy-proof if students have incentives to report their true preferences.

[5] See also Tang and Yu (2014) for an alternative algorithm for the EADAM.

but the students' individual characteristics in which those priorities are based. Second, in our case the resulting extended matching is an allocation of both school seats and students' characteristics. Third, the possible welfare gains we capture derive from the exchange of characteristics. That is, the resulting extended matching is justified with the final allocation of transferable characteristics. Fourth, all students that are relocated to a new school are strictly better off. However, the SETC considers exchanges of characteristics and, contrary to the stable improvement cycle algorithm in Erdil and Ergin (2008), some of the students that participate in these cycles only exchange their characteristics and facilitate other exchanges, and they are weakly better off. Finally, there is a technical difference. Our framework does not require the introduction of additional conditions on the set of priorities that may be ignored.[6] Our results only require that school priorities are monotonic in students' characteristics.

The rest of the paper is organized as follows. In Section 2, we introduce the model and notation. In Section 3, we present our main results that we prove in Section 4. In Section 5, we relate our framework of transferable characteristics to the school choice with consent proposed by Kesten (2010). In Section 6, we conclude.

2 Notation and Definitions

We present the standard school choice problem and then introduce the extended model with partially transferable characteristics.

Let I be a finite set of students and S a finite set of schools where students have to be allocated. Each student i is equipped with a strict preference P_i over $S \cup \{\varnothing\}$,[7] where $\{\varnothing\}$ stands for the option of being unassigned. We denote by R_i the weak preference relation associated to P_i defined in the standard way and by P a generic students' preference profile. Let $\mathcal{P}$ denote the set of all students' preference profiles. Each school s has a limited number of seats available q_s.

A ***matching*** is a function $\mu : I \to S \cup \{\varnothing\}$ such that (i) for each $i \in I$, $\mu(i) \in S \cup \{\varnothing\}$ and (ii) for each $s \in S$, $\#\mu^{-1}(s) \leq q_s$. A matching μ' ***Pareto dominates*** the matching μ if for each $i \in I$, $\mu'(i)\ R_i\ \mu(i)$, and for some $j \in I$, $\mu'(j)\ P_j\ \mu(j)$.

[6] Assumption 1 in Dur et al. (2019). See Remark 1 in Section 2.

[7] A strict preference is a complete, antisymmetric, and transitive binary relation.

The final component of the school choice problem is the priorities of schools. Each school ranks prospective students according to a priority order. Our contribution is to explore the structure of such priority orders. We consider that school priorities may depend on different characteristics of students. Some of those characteristics are intrinsic to each student, but some characteristics can be exchanged between students. The relevant priorities for schools depend on the final allocation of such characteristics.

For each student i let $\omega(i) = (\omega^s(i))_{s\in S}$ be the initial vector of transferable characteristics that influence the position of student i at each school. Each student initial endowment consists of transferable characteristics specific to each school. For each school s, let $\Omega^s = \cup_{i\in I}\omega^s(i)$ be s's set of available transferable characteristics. For each school s let λ^s be a bijection from students to Ω^s. That is, λ^s is a permutation of s's transferable characteristics between the students. For each $i \in I$, and $s \in S$ there is $j \in I$ with $\lambda^s(i) = \omega^s(j)$, and for each $j, j' \in I$, $\lambda^s(j) \neq \lambda^s(j')$. For each s, let $\mathcal{L}^s$ be the set of all permutations of s's transferable characteristics between students. We call $\lambda = (\lambda_s)_{s\in S}$ an ***allocation of transferable characteristics***. Finally, for each student i and each allocation λ, $\lambda(i) \equiv (\lambda^s(i))_{s\in S}$. We denote by ω the initial endowment allocation of transferable characteristics.

When the characteristics are transferable, the assignment of such characteristics is relevant. Note that each admissible $\lambda \in \mathcal{L}$ can be obtained via exchange cycles of characteristics between the students. An ***extended matching*** is a pair (μ, λ) such that μ is a matching and $\lambda \in \mathcal{L}$. Let $\mathcal{M}$ be the set of all extended matchings.

In the extended framework, school priorities do not compare only students, but pairs of students and the allocations of transferable characteristic that they present to the school choice process. Hence, each school is equipped with a complete, transitive, and antisymmetric binary relation $\succ_s$ on $I \times \Omega^s$. We use the notation $\succsim_s$ to refer to the weak priority relation associated to $\succ_s$.

Throughout this paper, we assume that transferable characteristics are monotonous in the sense that affect all the students in the same direction.

Monotonous Priorities For each i, j and s, for each $l, l' \in \Omega^s$: $(i, l) \succ_s (i, l')$ if and only if $(j, l) \succ_s (j, l')$.

Under monotonous priorities, for each s the set Ω^s is naturally ordered and, abusing notation, for each $\Lambda^s \subseteq \Omega^s$ we define

$$\max\{\Lambda^s\} \equiv \{l \in \Lambda^s, \text{ for each } i \in I, l' \in \Lambda^s, \, (i,l) \succsim_s (i,l')\}.$$

Remark 1. *Under monotonous priorities, for each school s, each $i_0, i_1, i_2, i_3 \in I$, and each extended priority $\succsim$, if*

$$\begin{gathered} \succ_s (i_1, \lambda^s(i_1)) \succ_s (i_2, \lambda^s(i_2)) \succ_s (i_3, \lambda^s(i_3)), \quad \textit{and} \\ (i_3, \max\{\lambda^s(i_0), \lambda^s(i_3)\}) \succsim_s (i_1, \lambda^s(i_1)) \end{gathered}$$

then $(i_3, \max\{\lambda^s(i_0), \lambda^s(i_3)\}) \succsim_s (i_2, \lambda^s(i_2))$.

Finally, we present the stability notion that takes into account the fact that school priorities depend on the identity of the students and some transferable characteristics.

An extended matching (μ, λ) is **(ex-post) stable** if:

- μ is λ-***fair***: for each $i, j \in I$, $\mu(j) \; P_i \; \mu(i)$ implies $(j, \lambda^s(j)) \succ_{\mu(j)} (i, \lambda^s(i))$.
- μ is ***individually rational***: for each $i \in I$, $\mu(i) \; R_i \; \{\varnothing\}$
- μ is ***not wasteful***: if for no $i \in N$ and $s \in S$, $s \; P_i \; \mu(i)$ and $\#\mu^{-1}(s) < q_s$.

The interpretation of *(ex-post) stable* coincides with the natural notion of stability. An *(ex-post) stable* extended matching does not generate complaints of students that would like to improve the school that they are assigned to. The matching proposed is justified with the final allocation of transferable characteristics.

It is worth to note that our notion of *(ex-post) stable* is parallel to *partial stability* in Dur et al. (2019) but we provide a rationale and structure to the admissible violations of the initial priorities. In the light of Remark 1, our extended priority structure does not call for the introduction of additional restrictions on the set of admissible violations of fairness as Assumption 1 in Dur et al. (2019).

We are interested in obtaining *(ex-post) stable* extended matching that are not Pareto dominated by other *(ex-post) stable* extended matchings. If there is no possibility of exchange of transferable characteristics, the students proposing deferred acceptance algorithm selects the student optimal stable extended matching (SOSM). The *(ex-post) stable* (μ, ω) is the ***student optimal stable extended matching (SOSM)*** if μ is not Pareto dominated by another (ex-post) stable extended matching (ν, ω).

When the students may exchange their transferable characteristics, we could find *(ex-post) stable* extended matchings (μ, λ) such μ Pareto dominates the SOSM matching. We focus on extended matchings that can be obtained by limited exchanges of transferable characteristics that lead to changes that justify the change of the students' match. Given an extended matching (μ, λ), we say $(\bar{\mu}, \bar{\lambda})$ is a ***reshuffle of*** (μ, λ) if for each $i \in I$, for each $s \notin \{\mu(i), \bar{\mu}(i)\}$, $\lambda^s(i) = \bar{\lambda}^s(i)$.

We are now in condition to present the notion that captures the idea of obtaining efficient matchings that are required to satisfy fairness and stability when transferable characteristics can be exchanged.

An extended matching (μ, λ) is ***constrained efficient*** if it is *(ex-post) stable* and for no *(ex-post) stable* reshuffle (μ', λ'), μ' Pareto dominates μ.

Example 1. *Let* $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, $q_{s_x} = 1$ *for* $x = 1, 2, 3$. *The students' preferences are:*

P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_1	s_1
s_1	s_2	s_2
s_3	s_3	s_3

*Each school uses two criteria to determine their priorities. They consider whether students have a sibling already enrolled at the school and whether they live in the walk-zone of the school. These criteria determine four coarse priority classes in each school. Each school prioritizes students with **Sibling+Walk-Zone**, and those students who have a **Sibling** but do not live in the **Walk-Zone** to students who live in its **Walk-Zone** with no enrolled **Sibling**. Finally, the inverse natural order breaks ties inside each priority class.*

Assume that no student has any sibling and walk-zone characteristics are transferable. Student i_1 *lives in school* s_1 *walk-zone, Student* i_2 *lives in school* s_2 *walk-zone, while* *student i_3 lives out of the school district. No student lives in s_3's walk-zone or has a sibling in s_3. Hence, we can write the initial endowment allocation of transferable characteristics:*

$$\begin{pmatrix} \omega(i_1) \\ \omega(i_2) \\ \omega_(i_3) \end{pmatrix} = \begin{pmatrix} (\omega^{s_1}(i_1), \omega^{s_2}(i_1), \omega^{s_3}(i_1)) \\ (\omega^{s_1}(i_2), \omega^{s_2}(i_2), \omega^{s_3}(i_2)) \\ (\omega^{s_1}(i_3), \omega^{s_2}(i_3), \omega^{s_3}(i_3)) \end{pmatrix} = \begin{pmatrix} (1,0,0) \\ (0,1,0) \\ (0,0,0) \end{pmatrix}.$$

Schools' priorities under the initial endowment allocation of transferable characteristics are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_1, 1)$	$(i_2, 1)$	$(i_3, 0)$
$(i_3, 0)$	$(i_3, 0)$	$(i_2, 0)$
$(i_2, 0)$	$(i_1, 0)$	$(i_1, 0)$

The SOSM for the initial endowment of transferable characteristics is (μ, ω) with $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$.

When students $\{i_1, i_2\}$ exchange their transferable characteristics, the allocation of exchangeable characteristics is

$$\begin{pmatrix} \lambda(i_1) \\ \lambda(i_2) \\ \lambda(i_3) \end{pmatrix} = \begin{pmatrix} (\lambda^{s_1}(i_1), \lambda^{s_2}(i_1), \lambda^{s_3}(i_1)) \\ (\lambda^{s_1}(i_2), \lambda^{s_2}(i_2), \lambda^{s_3}(i_2)) \\ (\lambda^{s_1}(i_3), \lambda^{s_2}(i_3), \lambda^{s_3}(i_3)) \end{pmatrix} = \begin{pmatrix} (0,1,0) \\ (1,0,0) \\ (0,0,0) \end{pmatrix},$$

and schools' extended priorities under λ are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_2, 1)$	$(i_1, 1)$	$(i_3, 0)$
$(i_3, 0)$	$(i_3, 0)$	$(i_2, 0)$
$(i_1, 0)$	$(i_2, 0)$	$(i_1, 0)$

The SOSM under λ is (μ', λ) with $\mu' = \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$. Clearly, μ Pareto dominates μ', and student i_3 has not justified envy for i_1 under the extended priorities obtained with the allocation of transferable characteristics λ.

3 Improvement Cycles for Extended Matchings

Our approach follows Erdil and Ergin (2008) and Dur et al. (2019) that propose a method for finding fair Pareto improving trade cycles upon SOSM for coarse priorities with arbitrary tiebreakers and partially non-enforceable priorities respectively. In both papers, the logic behind improving cycles is parallel. For an initial stable matching, if there's a vacant position at some school, that position may be assigned to one student such that no other student with higher priority at that school prefers that vacant position to her position at the initial matching. In our extended framework this rationale cannot be applied immediately. Although the students may be willing to accept any position at a desirable school, depending on the student that exchanges the transferable characteristic some violation of fairness may appear. For this reason, Pareto improvements involving two students may require the participation of additional students who just swap transferable characteristics without involving a change of school. Moreover, once a student leaves a position in a school, she may start a process similar to a vacancy chain (Blum et al., 1997). The first student in the priority ranking of the school may be admitted in the school without any need of the exchange in the transferable characteristics since the student that leaves the vacant position obtains a position at a preferred school.

Given an *(ex-post) stable* extended matching (μ, λ), for each school $j \in I$, let:

- $D_{(\mu,\lambda)}(j) = \{i \in I : \mu(j)\ R_i\ \mu(i)\}$ and $\tilde{D}_{(\mu,\lambda)}(j) = \{i \in I : \mu(j)\ P_i\ \mu(i)\}$.
- $X_{(\mu,\lambda)}(j) = \{i \in D_{(\mu,\lambda)}(j) : \forall k \in \tilde{D}_{(\mu,\lambda)}(j) \setminus \{i\}, (i, \max\{\lambda^s(i), \lambda^s(j)\}) \succ_s (k, \lambda^s(k))\}$.

The set $\tilde{D}_{(\mu,\lambda)}(j)$ contains all the students who prefer the match for student j rather than their own match. The set $D_{(\mu,\lambda)}(j)$ also includes all students who are matched to $\mu(j)$. The set $X_{(\mu,\lambda)}(j)$ includes all the students who would be willing to occupy j's position at $\mu(j)$ and there would not be any instance of envy if they are matched to $\mu(j)$ should j leave her position. The members of $X_{(\mu,\lambda)}(j)$ are those students in $D_{(\mu,\lambda)}(j)$ that either after they obtain $\lambda^{\mu(j)}(j)$ or maintaining $\lambda^{\mu(j)}(j)$ are ranked above the remaining members of $\tilde{D}_{(\mu,\lambda)}(j)$. Hence, if j moves to a preferred school and a member of $X_{(\mu,\lambda)}(j)$ gets j's position at $\mu(j)$, nobody could argue that the change violates her priority over $\mu(j)$.

Let $G = (V; E)$ be a directed graph with the set of vertices V, and the set of directed edges E, which is a set of ordered pairs of V .

For each extended matching (μ, λ), $G(\mu, \lambda) = (I; E(\mu, \lambda))$ is the (directed) application graph associated with (μ, λ) where the set of directed edges $E(\mu, \lambda) \subseteq I \times I$ is as follows: $ij \in E(\mu, \lambda)$ (that is, i points to j) if and only $i \in X_{(\mu,\lambda)}(j)$. A set of edges $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is a path if the vertices $i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}$ are distinct, and a cycle if the vertices $i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}$ are distinct and $i_1 = i_{n+1}$. A student i is involved in the cycle ϕ if there is a student j such that $ij \in \phi$. A cycle $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is solved when for each $ij \in \phi$, student i is assigned to $\mu(j)$ to obtain a new matching. Formally, we denote the solution of a cycle by the operation $\circ$ that is, $\eta = \phi \circ \mu$ if and only if for each $ij \in \phi$, $\eta(i) = \mu(j)$, and for each $i' \notin \{i_1, \ldots, i_n\}$ $\eta(i') = \mu(i')$. A cycle ϕ is an improvement cycle for $G(\mu, \lambda)$ if there is $ij \in \phi$ such that $i \in \tilde{D}_{(\mu,\lambda)}(j)$.

The following algorithm is built on an *(ex-post) stable* extended matching and is defined by solving cycles iteratively:

Student Exchange with Transferable Characteristics (SETC):

Step 0: Let (μ_0, λ_0) be an *(ex-post) stable* extended matching.

Step $k \geq 1$: Given an extended matching $(\mu_{k-1}, \lambda_{k-1})$,

$(k.1)$ if there is no improvement cycle in $G(\mu_{k-1}, \lambda_{k-1})$, then the algorithm terminates and $(\mu_{k-1}, \lambda_{k-1})$ is the matching obtained,

$(k.2)$ otherwise, solve one of the improvement cycles in $G(\mu_{k-1}, \lambda_{k-1})$ say ϕ_k let $\mu_k = \phi_k \circ \mu_{k-1}$, and define λ_k as follows. For each $i \in I$, let $s_k = \mu_k(i)$ and $s_0 = \mu_0(i)$.

- For each $s \notin \{s_0, s_k\}$, $\lambda_k^s(i) = \lambda_0^s(i)$.
- If there is no i' such that $ii' \in \phi_k$, then $\lambda_k^{s_k}(i) = \lambda_{k-1}^{s_k}(i)$.
- If there is i' such that $ii' \in \phi_k$ then $\lambda_k^{s_k}(i) = \max\{\lambda_{k-1}^{s_k}(i), \lambda_{k-1}^{s_k}(i')\}$.
- If there is j such that $\lambda_0^{s_0}(i) = \lambda_k^{s_0}(j)$, then $\lambda_k^{s_0}(i) = \lambda_0^{s_0}(j)$, otherwise $\lambda_k^{s_0}(i) = \lambda_0^{s_0}(i)$.

Since the sets of schools and students are finite, the algorithm stops in a finite number of steps. Actually, there are at most $\frac{1}{2}\#I(\#I-1)$ possible Pareto improvements. The definition provides a class of algorithms since there can be several incompatible Pareto improvements and the order in which cycles are solved may lead to different final outcomes.

The next example shows the relevance for constructing improvement cycles of students who do not strictly benefit from the exchange of transferable characteristics.

Example 2. *Let* $I = \{i_1, i_2, i_3, i_4\}$*,* $S = \{s_1, s_2, s_3\}$*,* $q_{s_x} = 1$ *for* $x = 1, 3$*; and* $q_{s_2} = 2$*. The students' preferences are:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_2	s_1	s_1	s_2
s_1	s_2	s_2	s_3
s_3	s_3	s_3	s_1

Each school uses two criteria to determine their priorities. Schools 1 *and* 3 *consider whether students have a sibling already enrolled at the school and whether they live in the walk-zone of the school. These criteria determine four coarse priority classes in each school. Each school prioritizes students with both characteristics* ***Sibling+Walk-Zone****, and those students who have a* ***Sibling*** *but do not live in the* ***Walk-Zone*** *to students who live in its* ***Walk-Zone*** *with no enrolled* ***Sibling****. Finally, the inverse natural order breaks ties inside each priority class. School* s_2 *orders students according to the outcome of the exam, using the* ***Walk-Zone*** *for breaking ties (and eventually with the inverse natural tiebreaker). Students* i_1 *and* i_2 *live in* s_1*'s walk-zone. Student* i_1 *has a sibling in* s_1 *but their parents would like to move their children to* s_2*. Student* i_4 *has the highest test-score overall for* s_2 *while students* i_1*,* i_2 *and* i_3 *have same test-score ranking. Finally, student* i_4 *lives in* s_2*'s walk-zone.*

$$\begin{pmatrix} \omega(i_1) \\ \omega(i_2) \\ \omega_(i_3) \\ \omega(i_4) \end{pmatrix} = \begin{pmatrix} (\omega^{s_1}(i_1), \omega^{s_2}(i_1), \omega^{s_3}(i_1)) \\ (\omega^{s_1}(i_2), \omega^{s_2}(i_2), \omega^{s_3}(i_2)) \\ (\omega^{s_1}(i_3), \omega^{s_2}(i_3), , \omega^{s_3}(i_3)) \\ (\omega^{s_1}(i_4), \omega^{s_2}(i_4), , \omega^{s_3}(i_4)) \end{pmatrix} = \begin{pmatrix} (1,0,0) \\ (1,0,0) \\ (0,0,0) \\ (0,1,0) \end{pmatrix}.$$

Schools' priorities under the initial endowment allocation of transferable characteristics are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_1, 1)$	$(i_4, 1)$	$(i_4, 0)$
$(i_2, 1)$	$(i_3, 0)$	$(i_3, 0)$
$(i_4, 0)$	$(i_2, 0)$	$(i_2, 0)$
$(i_3, 0)$	$(i_1, 0)$	$(i_1, 0)$

Moreover, $(i_3, 1) \succ_{s_1} (i_2, 1)$ *and* $(i_1, 1) \succ_{s_2} (i_2, 0)$*.*

The SOSM for the initial endowment of transferable characteristics is (μ, ω) *with* $\mu = \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_2)\}$*. The assignment* $\mu' = \{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_2)\}$ *Pareto dominates* μ *but if* i_1 *and* i_3 *exchange their transferable characteristics, then the resulting extended matching would result in justified envy since* $\omega^{s_2}(i_3) = 0$ *and* $(i_2, 0) \succ_{s_2} (i_1, 0)$*. However, if student* i_4 *participates in the exchange of characteristics, we would obtain the reshuffle* λ*:*

$$\begin{pmatrix} \lambda(i_1) \\ \lambda(i_2) \\ \lambda(i_3) \\ \lambda(i_4) \end{pmatrix} = \begin{pmatrix} (\lambda^{s_1}(i_1), \lambda^{s_2}(i_1), \lambda^{s_3}(i_1)) \\ (\lambda^{s_1}(i_2), \lambda^{s_2}(i_2), , \lambda^{s_3}(i_2)) \\ (\lambda^{s_1}(i_3), \lambda^{s_2}(i_3), \lambda^{s_3}(i_3)) \\ (\lambda^{s_1}(i_4), \lambda^{s_2}(i_4), \lambda^{s_3}(i_4)) \end{pmatrix} = \begin{pmatrix} (0,1,0) \\ (1,0,0) \\ (1,0,0) \\ (0,0,0) \end{pmatrix},$$

and the extended matching (μ', λ) *is* (ex-post) stable*.*

Starting from the initial SOSM, in Figure 1 we present the graph where each student points to the positions that students whose position would like to occupy (including indifference relations). In Figure 2 we show the strict improvements that would not generate justified envy and observe that no cycle can be constructed. Note that i_1 *does not point* i_3 *because* $(i_2, \omega_{i_2}^{s_2}) \succ_{s_2} (i_1, \max\{\omega_{i_1}^{s_2}, \omega_{i_3}^{s_2}\})$*. Finally, in Figure 3 and we present the graph associated to* (μ, ω)*. We observe the existence of a unique cycle* $\gamma = i_1 i_4 i_3 i_1$*. Solving* γ *generates the extended matching* (μ', λ)*. In Figure 4, we present the graph* $G(\mu', \lambda)$*. The graph contains no improvement cycle and indeed the extended matching* (μ', λ) *is* (ex-post) stable*.*

Remark 2. *The school priorities presented in Example 2 are consistent with point-system based priorities. Point-systems generate additively separable extended priorities. That is for each school* s*, for each pair of students* i_x, i_y *and each* $\lambda^s, \bar{\lambda}^s \in \Omega^s$*,* $(i_x, \lambda^s) \succ_s (i_y, \lambda^s)$ *if and only if* $(i_x, \bar{\lambda}^s) \succ_s (i_y, \bar{\lambda}^s)$*.*

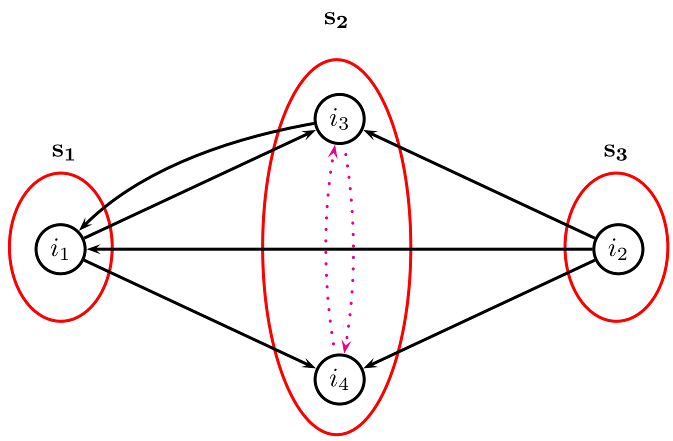

Figure 1: Example 2. Student i_x points student i_y if $i_x \in D_{(\mu,\omega)}(i_y)$. Solid lines: i_x points i_y if $i_x \in \tilde{D}_{(\mu,\omega)}(i_y)$. Dotted Lines: i_x points i_y if $i_x \in D_{(\mu,\omega)}(i_y)$, $\mu(i_x) = \mu(i_y)$.

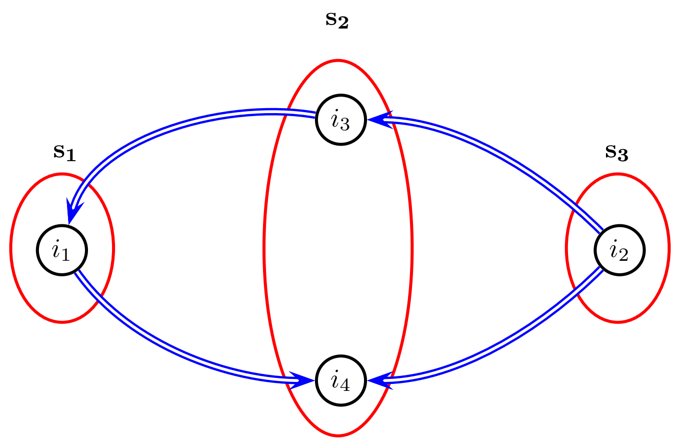

Figure 2: Example 2. Graph associated to (μ, ω). Student i_x points student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$ and $\mu(i_x) \neq \mu(i_y)$.

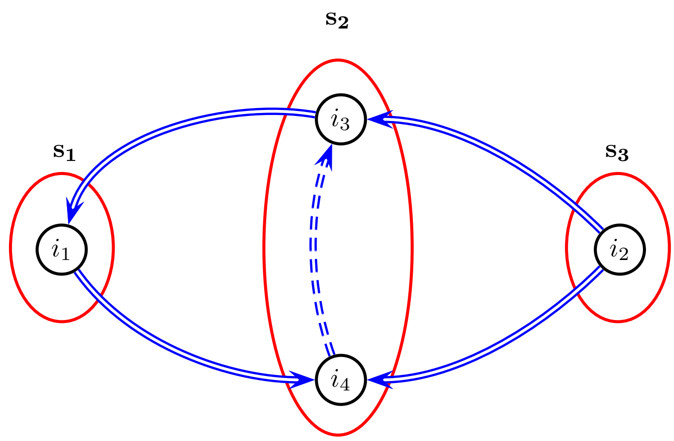

Figure 3: Example 2. $G(\mu, \omega)$. Student i_x points student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$.

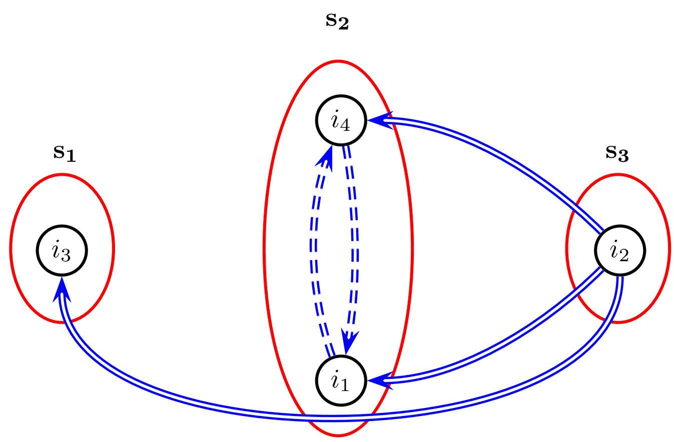

Figure 4: Example 2. $G(\mu', \lambda)$. Student i_x points student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$.

Next, we present our main result. It turns out that starting from any *(ex-post) stable* extended matching the application of an algorithm in the SETC class always yields a constrained efficient and *(ex-post) stable* extended matching. Moreover, any constrained efficient extended stable matching can be obtained from a SETC algorithm starting at the SOSM extended matching. Hence, the SETC class identifies all the improvement cycles that yield an *(ex-post) stable* extended matchings.

Theorem 1. *For each problem, an extended matching is constrained efficient and Pareto dominates the SOSM if and only if it is obtained by an algorithm within the SETC class starting with the SOSM extended matching.*

The proof of Theorem 1 is presented in the next section. The proof follows similar arguments to the proof of Dur et al. (2019) but the extended model generates important intricacies. Transferable characteristics differ between students and only exchanges involving specific students in a school may be mutually viable. Moreover, improvement cycles may need to involve students who do not strictly improve by the exchange but facilitate the reassignment by trading their transferable characteristics.

Two immediate consequences follow from Theorem 1. Since the result of a SETC is constrained efficient and *(ex-post) stable* with respect to the final allocation of transferable characteristics, then it is the result of the SOSM for the final allocation of transferable characteristics.

Corollary 1. *For each problem and each stable matching* μ_0 *and each algorithm in the SETC class, if the extended matching* (μ, λ) *is the outcome of the SETC algorithm then* (μ, λ) *is the SOSM with an initial endowment of transferable characteristics* λ*.*

We conclude this section analyzing the incentives of students to reveal their true preferences when the allocation of schools' seats is determined by an algorithm in the SETC class. For that purpose, we need to introduce further notation that relates the outcomes of different problems defined in different preference profiles.

A mechanism is a mapping $\Psi : \mathcal{P} \to \mathcal{M}$. The application of a SETC algorithm starting with the SOSM extended matching corresponding to each preference profile defines a mechanism that always selects a *(ex-post) stable* and constrained efficient extended matching. We call the class of such mechanisms as the ***students' optimal with transferable characteristics (SOTC)*** class of rules.

Strategy-proofness A mechanism Ψ satisfies ***strategy-proofness*** if for each $i \in N$, each $P, P' \in \mathcal{P}$ such that for each $j \neq i$ $P_j = P'_j$ with $\Psi(P) = (\mu, \lambda)$ and $\Psi(P') = (\mu', \lambda')$, $\mu(i) \; R_i \; \mu'(i)$.

By the results in Abdulkadiroğlu et al. (2009); Kesten (2010); Alva and Manjunath (2019); Kesten and Kurino (2019), since the matching selected by any SETC algorithm starting with the SOSM matching Pareto dominates the SOSM matching for the initial endowment of characteristics and it results in efficient allocations, any SETC is manipulable at some profile of students' preferences.

Proposition 1. *There is no mechanism in the SOTC class that satisfies* strategy-proofness*.*

Proof. Let A be an algorithm in the SETC, define the SOTC mechanism Ψ that for each profile of students' preferences selects the matching obtained through the application of A at that preference profile. By Theorem 1, for each preference profile the matching selected by Ψ is *(ex-post) stable* and Pareto efficient. For each $P \in \mathcal{P}$, Ψ selects an extended matching that represents a Pareto improvement upon the SOSM matching. By Abdulkadiroğlu et al. (2009), the SOSM is in the Pareto frontier of the set of mechanisms that satisfy stability and *strategy-proofness*. Hence, Ψ violates *strategy-proofness*. □

4 Proof of Theorem 1

Although Theorem 1 refers specifically to the application of SETC algorithms to the SOSM extended matching, the analysis can be carried out from any arbitrary *(ex-post) stable* extended matching. We study separately the proofs of necessity and sufficiency sides of the results.

4.1 Proof of "if" part

For a given problem $(R, \succsim)$ and a stable extended matching (μ_0, λ_0) consider an algorithm in the SETC class. Let K be the last step of the algorithm and (μ_k, λ_k) be the extended matching selected at $k \in \{1, \ldots, K-1\}$. A cycle is solved at each step of the algorithm, which implies that the students in the cycle are better off and no student is worse off at the new matching obtained by solving the cycle. Thus, the matching at each step *Pareto dominates* the matching in the previous step, and for each $k \geq 1$, if student j is not involved in any improvement cycle at Step k, $\tilde{D}_{(\mu_k,\lambda_k)}(j) \subseteq \tilde{D}_{(\mu_{k-1},\lambda_{k-1})}(j)$. Hence, if i points to j in $G(\mu_{k-1},\lambda_{k-1})$ and both students are not involved in an improvement cycle at Step k then i points to j in $G(\mu_k,\lambda_k)$.

Lemma 1. *Each extended matching obtained by a SETC algorithm is stable.*

Proof. Let $(\mu_k,\lambda_k\}$ be the extended matching obtained at Step $k \in \{0,\ldots,K-1\}$. We prove the result by induction on k. The initial extended matching (μ_0,λ_0) is stable.

Fairness Assume that $(\mu_{k-1},\lambda_{k-1})$ is fair. Take any pair of students (i,j) such that $\mu_k(j)\ P_i\ \mu_k(i)$. At each step of the algorithm, each student is either better off (she is in a solved cycle) or she is assigned to the same school as in the previous step. Let ϕ_k denote the improvement cycle solved in step k. Assume first that j is not involved in the cycle ϕ_k. Since $\mu_k(j)\ P_i\ \mu_k(i)$, $\mu_{k-1}(j)\ P_i\ \mu_{k-1}(i)$ and $i \in \tilde{D}_{(\mu_{k-1},\lambda_{k-1})}(j)$. Then, by fairness of $(\mu_{k-1},\lambda_{k-1})$, $(j,\lambda^{\mu_{k-1}(j)}(j)) \succ_{\mu_{k-1}} (i,\lambda^{\mu_{k-1}(j)}(i))$. Since j is not involved in ϕ_k, $\lambda^{\mu_{k-1}(j)}(j) = \lambda^{\mu_k(j)}(j)$. Since $i \in \tilde{D}_{(\mu_k,\lambda_k)}(j)$, $\lambda^{\mu_{k-1}(j)}(i) = \lambda^{\mu_k(j)}(i)$. Therefore $(j,\lambda^{\mu_k(j)}(j)) \succ_{\mu_k(j)} (i,\lambda^{\mu_k(j)}(i))$. Assume now that j is involved in ϕ_k. Let $j' \in I$ be such that $j'j \in \phi_k$. Hence, $\mu_{k-1}(j')\ P_i\ \mu_{k-1}(i)$, $i \in \tilde{D}_{(\mu_k,\lambda_k)}(j')$, and $\lambda^{\mu_{k-1}(j')}(i) = \lambda^{\mu_k(j')}(i)$. Since $j'j \in \phi_k$, $(j, \max\{\lambda_{k-1}^{\mu_{k-1}(j')}(j'), \lambda_{k-1}^{\mu_{k-1}(j)} \succ_{\mu_{k-1}(j')} (i,\lambda^{\mu_{k-1}(j')}(i))$, and $(j,\lambda^{\mu_k(j)}(j)) \succ_{\mu_k(j)} (i,\lambda^{\mu_k(j)}(i))$. Since i,j are arbitrary, (μ_k,λ_k) is *fair*.

Individual Rationality Since μ_0 is *individually rational*, and each student is never worse off after each step of the algorithm, the μ_K is *individually rational.*

Non-Wastefulness The initial match μ_0 is non wasteful. At each step students are assigned to better schools swapping their positions at schools, hence $\#\mu_k^{-1}(s)$ remains constant at any step of the algorithm. Assume school s has an empty slot at step k, then the school s has an empty slot at step 0. Since μ_0 is non wasteful and individually rational, for each student i with $\mu_0(i) \neq s$, $\mu_0(i)\ P_i\ s$. Since for each i, $\mu_k(i)\ R_i\ \mu_0(i)$, $\mu_k(i)\ R_i\ s$, and (μ_k,λ_k) satisfies non-wastefulness. □

Lemma 2. *For each stable extended matching (μ,λ) and $j \in I$, $X_{(\mu,\lambda)}(j) \subseteq \mu(\mu^{-1}(j))\backslash\{j\}$ if and only if $\tilde{D}_{(\mu,\lambda)}(j) = \{\varnothing\}$.*

Proof. If $\tilde{D}_{(\mu,\lambda)}(j) = \{\varnothing\}$, since $D_{(\mu,\lambda)}(j) = \mu(\mu^{-1}(j))$ and $X_{(\mu,\lambda)}(j) \subseteq D_{(\mu,\lambda)}(j)$, the result is immediate. On the other hand, if $\tilde{D}_{(\mu,\lambda)}(j) \neq \{\varnothing\}$, then by completeness and transitivity of schools' priorities there is $i \in \tilde{D}_{(\mu,\lambda)}(j)$ such that for each $i' \in \tilde{D}_{(\mu,\lambda)}(j)$, $(i, \lambda^{\mu(j)}(i)) \succsim_{\mu(j)} (i', \lambda^{\mu(j)}(i'))$. By monotonicity of priorities, $(i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(i)\}) \succ_{\mu(j)} (i, \lambda^{\mu(j)}(i))$. Therefore, $\mu(i) \neq \mu(j)$ and $i \in X_{(\mu,\lambda)}(j)$. $\square$

Lemma 3. *Let (μ, λ) and (η, λ') be* (ex-post) stable *extended matchings such that μ Pareto dominates η. For each $s \in S$, $\#\mu^{-1}(s) = \#\eta^{-1}(s)$.*

Proof. Let $N = \{i \in I : \mu(i)\ P_i\ \eta(i)\}$. Since μ Pareto dominates η, for each $j \in I \setminus N$, $\mu(j) = \eta(j)$. Consider school s and assume that $\#(N \cap \mu^{-1}(s)) > \#(N \cap \eta^{-1}(s))$. This implies that $\#\eta^{-1}(s) < q_s$. For each $i \in N \cap \mu^{-1}(s)$, $\mu(i) = s\ P_i\ \eta(i)$, which contradicts η non-wastefulness. Hence, $\#(N \cap \mu^{-1}(s)) \leq \#(N \cap \eta^{-1}(s))$. Finally, assume to the contrary there is s such that the strict inequality holds. Summing up the inequalities across schools, the number of students in N who are assigned to some school in matching η is larger than the number of students in N that are assigned to some school in matching μ. Hence there is a student $i \in N$ such that $\eta(i) \in S$, and $\mu(i) = \{i\}$. Since η is a individually rational matching, $\eta(i)\ P_i\ \mu(i)$ which contradicts the definition of N. $\square$

Lemma 4. *An extended matching obtained by an SETC algorithm is constrained efficient.*

Proof. Let (μ, λ) be an extended matching obtained by an SETC algorithm. By Lemma 1, (μ, λ) is *(ex-post) stable*. We show that there is no stable extended matching (ν, λ') such that ν Pareto dominates μ. Assume to the contrary, that (ν, λ') is a *(ex-post) stable* extended matching and ν dominates μ. By the definition of the SETC algorithms, there is no improvement cycle in the graph $G(\mu, \lambda)$. There are two cases:

Case 1. For each $j \in I$ $\tilde{D}_{(\mu,\lambda)} = \{\varnothing\}$. Then for each $\in I$, $X_{(\mu,\lambda)}(j) \subseteq \mu^{-1}(j) \setminus \{j\}$. Thus each student is assigned to her best school at μ and ν does not Pareto dominate μ

Case 2. There are chains in $G(\mu, \lambda)$ involving students who would like to change her assigned school, but there is no cycle. This implies that there are students who are only pointed by the students assigned to the same school.

Assume we are in Case 2. Since there is no improvement cycle, there is a set of students who are not pointed by any other student in $G(\mu, \lambda)$. Let $I_1 = \{i \mid \tilde{D}_{(\mu,\lambda)}(i) = \varnothing\}$. Let $i_1 \in I_1$ and $s_1 = \mu(i_1)$. Note that for each $j \in \mu(s_1)$, $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$ and $\mu(s_1) \subseteq I_1$ Since ν Pareto dominates μ, there does not exist any $j' \in I$, such that $\mu(j') \neq s_1$ and $\nu(j') = s_1$. Thus $\nu^{-1}(s_1) \subseteq \mu^{-1}(s_1)$. By Lemma 3, $\#\mu^{-1}(s_1) = \#\nu^{-1}(s_1)$ and we get

$\mu^{-1}(s_1) = \nu^{-1}(s_1)$. Since i_1 was arbitrary, this holds for each s such that $\mu^{-1}(s) \cap I_1 \neq \varnothing$.

Next, since there is no improvement cycle in $G(\mu, \lambda)$, then there is at least a student in $I \setminus I_1$ such that she is only pointed by students in I_1. Otherwise, there would be an improvement cycle or no improvement chains (Case 1). Let $I_2 = \{i \mid \tilde{D}_{(\mu,\lambda)}(i) \subseteq I_1\} \setminus I_1$ be the set of such students. Let $i_2 \in I_2$ and $s_2 = \mu(i_2)$. We first show that there is no j with $\mu(j) \neq s_2$ and $\nu(j) = s_2$. Assume to the contrary and since ν Pareto dominates μ, $s_2 \; P_j \; \mu(j)$ and thus, $j \in \tilde{D}_{(\mu,\lambda)}(i_2)$. Nevertheless, by definition i_2 is only pointed by students in I_1. By the previous paragraph, for each $j \in I_1$, $\mu(j) = \nu(j)$. Hence, $\nu^{-1}(s_2) \subseteq \mu^{-1}(s_2)$. By Lemma 3, $\#\mu^{-1}(s_2) = \#\nu^{-1}(s_2)$, and therefore $\mu^{-1}(s_2) = \nu^{-1}(s_2)$.

We can continue applying the same argument iteratively, to conclude that all students in any improving chain in $G(\mu, \lambda)$ have the same assignment under μ and ν. The students who are not in a chain in $G(\mu, \lambda)$, are contained in I_1 and have the same assignment in both μ and ν. We conclude that $\mu = \nu$ and ν does not Pareto dominate μ.

$\square$

4.2 Proof of the "only if" part

Let (μ_0, λ_0) a partially stable extended matching. We prove that each constrained efficient matching that Pareto dominates (μ_0, λ_0) can be obtained by an algorithm in the SETC class.

We use again the notion of improvement cycle we introduced in the previous subsection without making reference to the desirability graph. The following lemma is a crucial first step for the construction of improvement cycles in the desirability graph.

Lemma 5. *Let (μ, λ) and $(\nu, \bar{\lambda})$ be stable extended matchings such that ν Pareto dominates μ. Then there exists a set of disjoint improvement cycles $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ such that $\nu = \gamma_k \circ \ldots \circ \gamma_1 \circ \mu$, and there is λ'' obtained as in the definition of SETC such that (ν, λ'') is stable extended matching.*

Proof. Let $N \subseteq I$ be the set of students who strictly prefer their assignment under ν to the assignment under μ or such that $\lambda(i) \neq \lambda'(i)$. Partition the set N in three disjoint sets $N = N_1 \cup \; N_2 \cup N_3$. Define

$$N_1 \equiv \{i \in N \mid \mu(i) = \nu(i) \;\&\; \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\},$$
$$N_2 \equiv \{i \in M \mid \mu(i) \neq \nu(i) \;\&\; \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\},$$
$$N_3 \equiv \{i \in N \mid \mu(i) \neq \nu(i) \;\&\; \bar{\lambda}^{\nu(i)}(i) = \lambda^{\nu(i)}(i)\}.$$

Let $m = \#N$ and index the students in N in such that for each $j, j', j'' \in \{1, \dots, m\}$ $i_j \in N_1$, $i_{j'} \in N_2$, $i_{j''} \in N_3$ if and only if $j < j' < j''$. Let $\tilde{G}[(\mu, \lambda), (\nu, \lambda')] = (N, E)$ be a directed graph where the edges $E \subseteq N \times N$ are constructed in the following way:

- For each $i_j \in N_1$, i_j points l if and only if $\bar{\lambda}^{\mu(i_j)}(i_j) = \lambda^{\mu(i_j)}(l)$.
- For each $i_j \in N_2$, i_j points l if and only if $\bar{\lambda}^{\nu(i_j)}(i_j) = \lambda^{\nu(i_j)}(l)$.
- For each $i_j \in N_3$, i_j points an arbitrary student in $l \in N$ such that l has not been pointed by any $i_{j'}$ with $j' < j$ and $\mu(l) = \nu(i_j)$.[8]

In the graph $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda})]$, each student is pointed by a unique student and points to a unique student in N. Since N is finite, each student is in a cycle and no two cycles intersect. By construction, each of those cycles is an improvement cycle over μ and the extended matching $(\nu, \bar{\lambda})$ is obtained solving these cycles in any order. $\square$

Lemma 6. *Let* (μ, λ) *be an* (ex-post) stable *and* $(\nu, \bar{\lambda})$ *a* (ex-post) stable *reshuffle of* (μ, λ) *such that* ν *Pareto dominates* μ*, then there exists a sequence of cycles* $(\gamma_1, \dots, \gamma_k)$ *such that:*

- γ_1 *appears in* $G(\mu, \lambda)$.
- *for each* $k' \in \{2, \dots, k\}$, $\gamma_{k'}$ *in* $G(\gamma_{k'-1} \circ \dots \circ \gamma_1 \circ (\mu, \lambda))$,
- $\gamma_k \circ \gamma_{k-1} \circ \dots \circ \gamma_1 \circ (\mu, \lambda)$.

Proof. By Lemma 5, we can construct a set of improvement cycles $\Phi = \{\phi_1, \dots, \phi_q\}$. The result is trivial in the case where all the cycles in Φ appear in $G(\mu, \lambda)$: it follows that there are disjoint cycles in $G(\mu, \lambda)$ and solving them in any order leads to ν and to some λ' such that (ν, λ') is an *(ex-post) stable* reshuffle of (μ, λ). To prove the alternative case, we assume that none of the cycles in ϕ appears in $G(\mu, \lambda)$. This assumption is without loss

[8]Note that since $(\nu, \bar{\lambda})$ is a reshuffle of (μ, λ) such a student l exists for each $i_j \in N_3$.

of generality because of the following observation. If a cycle $\phi \in \Phi$ appears in $G(\mu, \lambda)$, then this cycle is solved first and $\mu' = \phi \circ \mu$ is obtained. If another cycle $\phi' \in \Phi$ also appears in $G(\mu', \lambda^*)$, by the fact that all the cycles in Φ are disjoint and that if there are two students forming a link in $G(\mu, \lambda)$,and those students do not belong to ϕ, then the link also appears in $G(\mu', \lambda^*)$. Following this logic, whenever a subset of cycles Φ appear in $G(\mu, \lambda)$, these cycles are solved first, and we focus on the case where none of the improvement cycles appear in $G(\mu, \lambda)$.

To show the existence of a cycle in $G(\mu, \lambda)$ first we prove that for any $\phi \in \Phi$ and any $ij \in \phi$, there exists some $k \in I$ such that $kj \in G(\mu, \lambda)$ and $lk \in \phi'$ for some $l \in I$ and $\phi' \in \Phi$. Consider an arbitrary $\phi \in \Phi$ and $ij \in \phi$.

- if $i \in X_{(\mu,\lambda)}(j)$, then $ij \in G(\mu, \lambda)$ by construction. Moreover, i is a part of ϕ, which implies there exists $l \in I$ with $li \in \phi$.

- if $i \notin X_{(\mu,\lambda)}(j)$, there exists a student i' such that $i' \in \tilde{D}_{(\mu,\lambda)}(j)$ and $(i', \lambda^{\mu(j)}(i')) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (i, \lambda^{\mu(j)}(i))$. Let k be , between those students, one such that $(k, \max\{\lambda^{\mu(j)}(k), \lambda^{\mu(j)}(j)\}) \succ_{\mu(j)} (k', \max\{\lambda^{\mu(j)}(k'), \lambda^{\mu(j)}(j)\})$ for each $k' \in D_{\mu,\lambda)}(j)$.[9] Note that $k \in X_{(\mu,\lambda)}(j)$, and therefore $kj \in G(\mu, \lambda)$. Finally, we check that k is in an improvement cycle in Φ. That is, there is $\phi' \in \Phi$ such that $lk \in \phi'$ for some $l \in I$. Assume to the contrary that $\mu(k) = \nu(k)$, and $\mu(j)\ P_k\ \mu(k) = \nu(k)$. Note that $k \in X_{(\mu,\lambda)}(j)$, $i \notin X_{(\mu,\lambda)}(j)$, $\nu(i) = \mu(j)$, and $\bar{\lambda}^{\mu(j)}(i) = \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}$). Since $(k, \lambda^{\mu(j)}(k)) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$, this is a contradiction, since $(\nu, \bar{\lambda})$ is *(ex-post) stable.* Thus, $\nu(k)\ P_k\ \mu(k)$, which implies that k is in an improvement cycle in Φ.

Thus, for any student j who is in an improvement cycle $\varphi \in \Phi$, there exists another student k such that $kj \in G(\mu, \lambda)$ and k is in an improvement cycle $\phi' \in \Phi$. Since the set of students in improvement cycles is finite, and each student is pointed at least by another student in N, and there exists a cycle γ_1 in $G(\mu, \lambda)$. Note that for each $ij \in \phi$ such that $ij \notin \gamma_1$, then $ij \notin G(\mu, \lambda)$, and $i \notin X_{(\mu,\lambda)}(j)$.

We next show that the matching $\gamma_1 \circ \mu$ Pareto dominates μ and it is weakly Pareto dominated by ν. Since $\gamma_1 \circ \mu$ solves a cycle in $G(\mu, \lambda)$ clearly $\gamma_1 \circ \mu$ Pareto dominates μ. Hence, we focus on proving that ν (weakly) Pareto dominates $\gamma_1 \circ \mu$. For any $kj \in \gamma_1$ such that $(\gamma_1 \circ \mu)(k) \neq \mu(k)$ note that $(\gamma_1 \circ \mu)(k) = \mu(j)$.

[9]By our definition of extended priorities the existence of such a student k is ensured. See Remark 1.

- If $kj \in \phi$ for some $\phi \in \Phi$, then $\nu(k) = \mu(j)$.
- If $kj \notin \phi$ for any $\phi \in \Phi$, we claim that $\nu(k)\ R_k\ \mu(j)$. Suppose that $\mu(j)\ P_k\ \nu(k)$, that is, $k \in \tilde{D}_{(\nu,\bar{\lambda})}(j)$. Consider the student $i \in I$ such that $ij \in \phi$ for some $\phi \in \Phi$, so $\nu(i) = \mu(j)$. By the definition of γ_1, $ij \notin G(\mu, \lambda)$. implies $\bar{\lambda}^{\mu(j)}(i) = \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$. Since $kj \in \gamma_1$, $kj \in G(\mu, \lambda)$ and $ij \notin G(\mu, \lambda)$, $(k, \lambda^{\mu(j)}(k)) \succ_{\mu(k)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$, which is a contradiction because $(\nu, \bar{\lambda})$ is *(ex-post) stable.*

Thus, under the matching $\gamma_1 \circ \mu$, each student in γ_1 is better off than under the matching μ and worse off than under the matching ν. Each remaining student is assigned to the same school to which she's assigned under μ which implies that the matching $\gamma_1 \circ \mu$ Pareto dominates μ and is weakly Pareto dominated by ν. Let λ_1 be the allocation of characteristics obtained by solving the cycle γ_1 according to the definition of the SETC algorithm. By the arguments in Lemma 1, $(\gamma_1 \circ \mu, \lambda_1)$ is *(ex-post) stable.* If the extended matching $(\gamma_1 \circ \mu)$ is equivalent to ν the proof is complete. If not we can use the same argument inductively. By Lemma 6, there is a set of distinct improvement cycles, such that the matching ν is obtained by solving these cycles over $\gamma_1 \circ \mu$ solving at each stage a cycle that appears in the graph defined by the SETC algorithm. □

5 Fully Transferable Characteristics

Theorem 1 is a general result without any reference to the construction of school priorities. In this section we discuss the implications for specific definitions of priorities. In the case that the transferable characteristics determine completely school priorities, when a student participates in an improvement cycle, it is equivalent to the fact of giving up completely the priorities the student have for a position at that school. With that intuition in mind, we propose a restricted domain of priorities that are completely defined by the transferable characteristics.

Fully Transferable Extended Priorities. For each $i, i', j, j'j \in I$ and $s \in S$, for each $\lambda^s, \bar{\lambda}^s \in \mathcal{L}^s$: $(i, \lambda^s) \succ_s (i', \bar{\lambda}^s)$ if and only if $(j, \lambda^s) \succ_s (j', \bar{\lambda}^s)$.

Under fully transferable priorities, the analysis of the algorithms in the SETC is simpler, since any student that desires the position of another student can obtain it with the exchange of the transferable characteristics. Under fully transferable priorities, the application of any SETC algorithm starting from the SOSM will stop at a constrained efficient extended matching. On the other hand, any Pareto improvement cycle starting at a *(ex-post) stable* extended matching would not generate any violation of fairness when priorities are fully transferable. Therefore, every Pareto efficient extended matching that improves upon the SOSM can be obtained as the outcome of a SETC algorithm.[10]

We devote the rest of this section to present an attractive subclass of SETC algorithms. Before the formal definition of the algorithms, it's worth to analyze the structure of the application graph G when priorities are fully transferable.

Lemma 7. *Let (μ, λ) be an* (ex-post) stable *extended matching and $G(\mu, \lambda)$ the (directed) application graph associated with (μ, λ). If schools' extended priorities are fully transferable and $i \in \tilde{D}_{(\mu,\lambda)}(j)$, then $ij \in G(\mu, \lambda)$.*

Proof. Let $s = \mu(j)$. Since $\in \tilde{D}_{(\mu,\lambda)}(j)$, $s \; P_i \; \mu(i)$. Since (μ, λ) is *(ex-post) stable*, for each $j' \neq i$ such that $s \; P_{j'} \; \mu(j')$, $(j, \lambda^s(j)) \succ_s (j', \lambda^s(j'))$. Therefore, since s extended priorities are fully transferable, $(i, \max\{\lambda^s(i), \lambda^s(j)\}) \succ_s (j', \lambda^s(j'))$ and $i \in X_{(\mu,\lambda)}(j)$. □

The previous lemma implies that under fully transferable priorities improvement cycles do not need the participation of students who transfer their characteristic but remain assigned to the same school.

At this point, before we introduce a specific selection of cycles in SETC algorithms, we need additional notation.

Let (μ, λ) be an extended matching. Define the graph $\tilde{G}(\mu, \lambda) \subset G(\mu, \lambda)$ as the restriction of $G(\mu, \lambda)$ where students only point to students assigned to different schools. Hence, $E = I$ and V are such that $ij \in \tilde{G}(\mu, \lambda)$ if and only if $i \in \tilde{D}_{(\mu,\lambda)}(j)$ and $i \in X_{(\mu,\lambda)}(j)$.

Let $T_0(\mu) = I$ and recursively for each $k \geq 1$

$$B_k(\mu) = \{i \in T_{k-1}(\mu) \mid \text{ for each } j \in T_{k-1}(\mu), \; \mu(i) \; R_i \; \mu(j)\},$$

and $T_k(\mu) = T_{k-1}(\mu) \setminus B_k(\mu)$. Let k^* be the smallest integer such that $B_{k^*}(\mu) = \{\varnothing\}$, and $T(\mu) \equiv T_{k^*}$. The set $B_1(\mu)$ contains the students that are assigned to their preferred

[10] This fact implies that the set of all possible matching obtained with the application of a SETC algorithm starting from the SOSM is the set α-fair matchings as defined by Alcalde and Romero-Medina (2017). In particular, the EADAM when all the students consent on waiving their priorities at every school is the result of a SETC algorithm (Kesten, 2010).

school at μ. Hence for each $j \in B_1(\mu)$ there is not j' such that $jj' \in \tilde{G}(\mu,\lambda)$. Recursively, for each $k < k*$ and $j \in B_k(\mu)$, $jj' \in \tilde{G}(\mu,\lambda)$ implies that $j' \in B_{k'}(\mu)$ for some $k' < k$. This implies that students in $I \backslash T(\mu)$ cannot participate in any Pareto improvement cycle. Moreover, if $T(\mu) = \{\varnothing\}$, then the extended matching (μ,λ) is Pareto efficient. Finally, for each extended matching (μ,λ) define the graph $G'(\mu,\lambda) \subset \tilde{G}(\mu,\lambda)$, such that each $ij \in G'(\mu,\lambda)$ if and only if $i, j \in T(\mu)$, $ij \in \tilde{G}(\mu,\lambda)$, and for each $j' \in T(\mu)$, $\mu(j)\ R_i\ \mu(j')$.

Lemma 8. *Let schools' extended priorities be fully transferable. If $T_k(\mu) \neq \{\varnothing\}$, then there is a Pareto Improvement cycle $\phi \in G'(\mu,\lambda)$ such that for each $ij \in \phi$, and each $j' \in T(\mu)$, $\mu(j)\ R_i\ \mu(j')$.*

Proof. Note that for each $i \in T(\mu)$ there is $j \in T(\mu)$ such that $i \in \tilde{D}_{(\mu,\lambda)}(j)$ and by the previous lemma, $i \in X_{(\mu,\lambda)}(j)$. Note that each $i \in T(\mu)$ points to at least some $j \in T(\mu)$ such that $\mu(i) \neq \mu(j)$. Since $T(\mu)$ is finite, there is at least one cycle in $G'(\mu,\lambda)$. $\square$

Lemma 9. *Let schools' extended priorities be fully transferable. If there are a pair of improvement cycles ϕ, ϕ' in $G'(\mu,\lambda)$ such that $\phi \subset \phi'$, then $\phi' \setminus \phi$ is an improvement cycle in $G'(\mu,\lambda)$*

Proof. Since ϕ and ϕ' are improvement cycles and priorities are fully transferable, there is $i \in \phi$ and j, k such that $ij \in \phi$, $ik \in \phi'$, and $i' \in \phi' \setminus \phi$ such that $i'j \in \phi'$. Since $ij \in \phi$ and$ik \in \phi'$, then $\mu(j) = \mu(k)$. On the other hand, $i'j \in \phi'$ and $\mu(j) = \mu(k)$ implies that $i' \in X_{(\mu,\lambda)}(k)$. Hence, $\phi' \setminus \phi$ is a improvement cycle in $G'(\mu,\lambda)$. $\square$

The previous lemmas show that under fully transferable characteristics the logic of the TTCM can be applied to find *(ex-post) stable* Pareto improvement cycles. This logic allows us to define a subclass of SETC algorithms.

Top Trade SETC Algorithms:

Step 0: Let (μ_0, λ_0) be a stable extended matching.

Step $k \geq 1$: Given an extended matching $(\mu_{k-1}, \lambda_{k-1})$,

(k.1) if there is no improvement cycle in $G(\mu_{k-1}, \lambda_{k-1})$, then the algorithm terminates and $(\mu_{k-1}, \lambda_{k-1})$ is the matching obtained,

(k.2) otherwise, solve one of the improvement cycles in $\phi \in G'(\mu_{k-1}, \lambda_{k-1})$, and let $\mu_k = \phi_k \circ \mu_{k-1}$, and λ_k be defined correspondingly to the definition of SETC.

It is worth to note that the Top Trade SETC Algorithms are a family of algorithms. While Lemma 9 implies that the selection of maximal improvement cycles is not an issue for the algorithm, it may be the case that several improvement cycles share some students. Note that under fully transferable priorities, in $G'(\mu, \lambda)$ each student points to all the students assigned to her best preferred school. Therefore, every selection rule for mutually incompatible improvement cycles generates a different algorithm and different potential outcomes.

In the next result, we prove that the application of the Top Trade SETC starting at the SOSM with the initial endowment of transferable characteristics yields the outcome of the sequential application of the TTCM on the SOSM matching,

Theorem 2. *Let μ_0 be the SOSM and (μ, λ) an outcome of and Top Trade SETC algorithm under μ_0, then μ is the matching obtained as an outcome of Gale's TTCM under the initial allocation of seats μ_0.*

Proof. By Lemmas 7 and 8, under monotonous and fully transfer preferences, the Top Trade SETC algorithm is well defined. At each stage of the algorithm, there is a group of students who obtain a seat at their best preferred available school till the stage where no Pareto improvement is possible, therefore μ is the outcome obtained after the application of the TTCM from the initial matching μ_0. If the initial matching is the SOSM, the stable matching selected by the Top Trade SETC coincides with the application of the TTCM after the selection of the SOSM. □

We conclude this section with a brief discussion relating the outcome of a Top Trade SETC and EADAM (Kesten, 2010). The motivation behind the EADAM is to explore the source of inefficiency of the SOSM due to fairness constraints and improve it on the efficiency dimension. An important observation made by Kesten (2010) is that the priority of student i at school s might not help her to get a better school under the SOSM at all. If this is the case, giving i the lowest priority at s instead of her current priority would not change her assignment and the DA would possibly select a matching that Pareto dominates the SOSM with the original priorities. Motivated by this observation, Kesten (2010) introduces the EADAM in a setting that allows students to consent for the violation of their own priorities. Clearly, our frameworks are different since we provide the foundations for a weaker notion of fairness, while Kesten (2010) considers the possibility of students dispensing with their priority rights. On the other hand, when priorities are fully transferable fairness restrictions that prevent mutually profitable exchange of assignments disappear as they do when all students consent. The structure of the EADAM (Kesten, 2010; Tang and Yu, 2014) implies that the consent of students allows for generating the best exchange available of schools' seats between students with respect the SOSM. Thus, from Theorem 2 we obtain the following corollary.

Corollary 2. *When priorities are fully transferable, the EADAM matching can be obtained with the application of a Top Trade SETC algorithm starting with the SOSM extended matching.*

6 Conclusions

In this paper, we generalize the school choice problem by defining school priorities on (transferable) students' characteristics. We define a family of algorithms– Student Exchange with Transferable Characteristics (SETC) class– that starting at a *(ex-post) stable* extended matching produce an *(ex-post) stable* extended matching that is not Pareto dominated by another *(ex-post) stable* extended matching. Moreover, any constrained efficient extended matching that Pareto improves upon a stable extended matching can be obtained via an algorithm in the SETC class. Finally, we show that an algorithm in the SETC class selects the outcome obtained with the iterated application of SOSM and TTCM when all students' characteristics are transferable.

Although the focus of this work has been on the application to school choice, there are further natural applications of the model. Recent research on the allocation of medical resources under triage has proved the possibilities of encompassing ethical values with the fair allocation of a single scarce resource by reserving part of the capacity to some groups of individuals (Pathak et al., 2020) Our work provides techniques that allow for Pareto improvements on fair allocations, when there is more than one type of object, ethical considerations may be relaxed, and transfers of characteristics are allowed.

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